# Variable Separation Solutions for the (2+1)-Dimensional Breaking Soliton Equation

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With the variable separation approach and based on the general reduction theory, we successfully obtain the variable separation solutions for the (2+1)-dimensional breaking soliton equation using of the projective Riccati equation. Based on one of the variable separation solutions and by selecting appropriate functions, a new type of interaction between the multi-valued and the single-valued solitons, that is a compacton-like semi-foldon and a 4-compacton, is investigated.

*Key words:* Breaking Soliton Equation; Projective Riccati Equation; Variable Separation Solution; Soliton.

## 1. Introduction

In recent decades, there has been noticeable progress in the study of the soliton theory. It is widely applied in many natural sciences such as chemistry, biology, mathematics, communication, and particularly in almost all branches of physics like fluid dynamics, plasma physics, field theory, nonlinear optics, condensed matter physics. Therefore, constructing possible exact solutions to a nonlinear model arising from the field of mathematical physics is a popular topic, although solving nonlinear physics problems is much more difficult than solving linear ones.

The extension of the variable separation approach in linear physics to nonlinear field has been a highlight, and three kinds of variable separation procedures have been presented recently. The first method is called the formal variable separation approach (FVSA) [1] or, equivalently, the symmetry constraints [2]. The second one is the procedure of searching the functional separable solutions (FSSs) [3] and the derivative-dependent functional separable solutions (DDFSSs) [4] to a partial differential equation (PDE) via the generalized conditional symmetries (GCSs) [5]. These two kinds of solutions have been discussed from the point of symmetry groups and the form ansatz. And they are the extensions of the classical additive or product separable solutions. The last (third) one is now called the multilinear variable separation approach (MLVSA) established first in 1996 for the Davey-Stewartsen (DS) equation [6]. Then the MLVSA has been revised and developed recently for various (2+1)-dimensional models, including the Broer-Kaup-Kupershmidt (BKK) system, the Nizhnik-Novikov-Veselov (NNV) equation, the long-wave-short-wave resonance interaction equation, the (2+1)-dimensional dispersive long wave (DLW) equation, and so on. Furthermore, Tang et al. [7] presented an universal formula for (2+1)-dimensional systems. Based on this universal formula, quite rich localized excitations, such as dromions, ring solitons, peakons, compactons, foldons, chaotic patterns and fractal patterns, can be obtained.

Now a significant and interesting issue is whether there are other direct and simple approaches to obtain variable separation solutions for (2+1)-dimensional systems. Recently, along with the variable separation idea, Zheng et al. [8, 9] realized variable separations for the BKK system and DLW equation by the mapping method which is successfully applied to derive rich traveling wave solutions [10]. However, one can find that it is too difficult to extend this mapping method to obtain variable separation solutions for other (2+1)-dimensional systems. Fortunately, to our excitement, with the help of an auxiliary equation and based on the general reduction theory, we successfully obtained five couples of variable separation solutions for the (2+1)-dimensional breaking soliton (BS) equation [11]

$$u_t + ku_{xxy} + 4kuv_x + 4ku_xv = 0, \tag{1}$$

$$u_y - v_x = 0, \tag{2}$$

where $u \equiv u(x,y,t), v \equiv v(x,y,t)$, and $k$ is an arbitrary constant. (1) and (2) describe the interaction of a Riemann wave propagating along the $y$-axis with a long wave along the $x$-axis and it seems that it has been investigated extensively where overlapping solutions have been derived. Radha and Lakshmanan [12] have proved that (1) and (2) possess the Painlevé property and found dromion-like structures. Zhang et al. [13] have derived a special variable separation solution of (1) and (2) with the help of the Bäcklund transformation.

It is shown in Section 2 that the novel exact solutions, including multiple soliton solutions, periodic soliton solutions and Weierstrass function solutions for the (2+1)-dimensional BS system, are derived by a special mapping transformation procedure. Section 3 deals with the nonlinear coherent structures of this system. That is, a new type of inelastic interaction between the multi-valued and the single-valued solitons is investigated. The last section consists of a short summary and discussion.

## 2. Variable Separation Solutions for the (2+1)-Dimensional BS Equation

Letting $f \equiv f(\xi(X)), g \equiv g(\xi(X))$ [where $\xi \equiv \xi(X)$ is a still undetermined function of the independent variables $X \equiv (x_0 = t, x_1, x_2, \cdots, x_m)$], the projective Riccati equation [14, 15] is defined by

$$f' = pfg, g' = q + pg^2 - rf, \tag{3}$$

where $p^2 = 1$, $q$ and $r$ are two real constants. When $p = -1$ and $q = 1$, (3) reduces to the coupled equations given in [14]. The following relation between $f$ and $g$ can be satisfied:

$$g^2 = -\frac{1}{p}\left[q - 2rf + \frac{r^2+\delta}{q}f^2\right], \tag{4}$$

where $\delta = \pm 1$. Equation (3) has ever been discussed in [15]. In this paper, we discuss some other cases.

**Lemma.** If the condition of (4) holds with other choices of $\delta$, the projective Riccati equation (3) has the following solutions:

(a) If $\delta = -r^2$, the Weierstrass elliptic function solution is admitted:

$$f = \frac{q}{6r} + \frac{2}{pr}\wp(\xi), g = \frac{12\wp'(\xi)}{q + 12p\wp(\xi)}. \tag{5}$$

Here $p = \pm 1$, the Weierstrass elliptic function $\wp(\xi) = \wp(\xi; g_2, g_3)$ satisfies $\wp'^2(\xi) = 4\wp^3(\xi) - g_2\wp(\xi) - g_3$, and $g_2 = \frac{q^2}{12}, g_3 = \frac{pq^3}{216}$.

(b) If $\delta = -\frac{r^2}{25}$, the projective Riccati equation has the Weierstrass elliptic function solution

$$f = \frac{5q}{6r} + \frac{5pq^2}{72r\wp(\xi)}, g = -\frac{q\wp'(\xi)}{\wp(\xi)(12\wp(\xi) + pq)}, \tag{6}$$

where $p = \pm 1$. Both $q$ and $r$ in (5) and (6) are arbitrary constants.

(c) If $\delta = h^2 - s^2$ and $pq < 0$, (3) has the solitary solution

$$f = \frac{q}{r + s\cosh(\sqrt{-pq}\xi) + h\sinh(\sqrt{-pq}\xi)},$$
$$g = -\frac{\sqrt{-pq}}{p}\frac{s\sinh(\sqrt{-pq}\xi) + h\cosh(\sqrt{-pq}\xi)}{r + s\cosh(\sqrt{-pq}\xi) + h\sinh(\sqrt{-pq}\xi)}, \tag{7}$$

where $p = \pm 1$; $s$ and $h$ are arbitrary constants.

(d) If $\delta = -h^2 - s^2$ and $pq > 0$, we have the trigonometric function solution

$$f = \frac{q}{r + s\cos(\sqrt{pq}\xi) + h\sin(\sqrt{pq}\xi)},$$
$$g = \frac{\sqrt{pq}}{p}\frac{s\sin(\sqrt{pq}\xi) - h\cos(\sqrt{pq}\xi)}{r + s\cos(\sqrt{pq}\xi) + h\sin(\sqrt{pq}\xi)}, \tag{8}$$

where $p = \pm 1$; $s$ and $h$ are arbitrary constants.

(e) If $q = 0$, (3) has the rational solution

$$f = \frac{2}{pr\xi^2 + C_1\xi - C_2},$$
$$g = -\frac{2pr\xi + C_1}{(pr\xi^2 + C_1\xi - C_2)p}, \tag{9}$$

where $C_1, C_2$, and $r$ are arbitrary constants, and $p = \pm 1$.

We now introduce the mapping approach via the above projective Riccati equation. The basic idea of the algorithm is: Considering a nonlinear partial differential equation (NPDE), with independent variables

$X \equiv (x_0 = t, x_1, x_2, \cdots, x_m)$ and the dependent variable $u \equiv u(X)$,

$$P(u, u_t, u_{x_i}, u_{x_i x_j}, \cdots) = 0, \tag{10}$$

where $P$ is a polynomial function of its arguments and the subscripts denote the partial derivatives, we assume that its solution is written as the standard truncated Painlevé expansion

$$u = A_0(X) + \sum_{i=1}^{n} (A_i(X) f(\xi(X)) + B_i(X) g(\xi(X))) f^{i-1}(\xi(X)). \tag{11}$$

Here $A_0(X), A_i(X), B_i(X)$ $(i = 1, \ldots, n)$ are arbitrary functions to be determined, and $f, g$ satisfy the projective Riccati equation (3).

To determine $u$ explicitly, one proceeds as follows: First, similar to the usual mapping approach, we can determine $n$ by balancing the highest-order nonlinear term with the highest-order partial derivative term in (10). Second, substituting (11) with (3) and (4) into the given NPDE, collecting the coefficients of the polynomials of $f^i g^j$ $(i = 0, 1, \cdots, j = 0, 1)$ and eliminating each of them, we can derive a set of partial differential equations of $A_0(X), A_i(X), B_i(X)$ $(i = 1, \ldots, n)$ and $\xi(X)$. Third, to calculate $A_0(X), A_i(X), B_i(X)$ $(i = 1, \ldots, n)$ and $\xi(X)$, we solve these partial differential equations. Finally, substituting $A_0(X), A_i(X), B_i(X)$ $(i = 1, \ldots, n), \xi(X)$, and the solutions (5)–(9) into (11), one obtains solutions of the given NPDE.

Now, we apply the above mapping approach to the (2+1)-dimensional BS system. According to the balancing procedure, (11) becomes

$$u = A + Bf(\xi) + Cf^2(\xi) + Dg(\xi) + Ef(\xi)g(\xi), \tag{12}$$

$$v = a + bf(\xi) + +cf^2(\xi) + dg(\xi) + ef(\xi)g(\xi), \tag{13}$$

where $A, B, C, D, E, a, b, c, d, e$ and $\xi$ are arbitrary functions of $\{x, y, t\}$ to be determined. Substituting (12), (13) with (3) and (4) into (1) and (2), collecting the coefficients of the polynomials of $f^i g^j$ $(i = 0, 1, 2, 3, 4, j = 0, 1)$ and setting each of the coefficients to zero, we can derive a set of partial differential equations for $A, B, C, D, E, a, b, c, d, e$ and $\xi$. It is difficult to obtain the general solution of these algebraic equations based on the solutions of (3). Fortunately, in the special case that $\xi = \chi(x) + \varphi(y, t)$, where $\chi(x)$, $\varphi(y, t) = \varphi(Hy + St)$ ($H$ and $S$ are two arbitrary constants) are two arbitrary variable separated functions of $x$ and $(y, t)$, respectively, we can obtain solutions of (1) and (2).

**Theorem.** For the (2+1)-dimensional BS system (1), (2), there are five couples of variable separated solutions, related to the projective Riccati equation (3).

(a) For $\delta = -r^2$, the Weierstrass elliptic function solution is

$$u = -\frac{1}{4k\chi_x\varphi_y}\big(k\chi_{xxx}\varphi_y + \chi_x\varphi_t - pqk\chi_x^3\varphi_y + 3pkr\chi_x^3\varphi_y f(\xi) + 3pk\chi_x\chi_{xx}\varphi_y g(\xi)\big),$$

$$v = -\frac{3}{4}pr\chi_x\varphi_y f(\xi), \tag{14}$$

where $p = \pm 1$, and $f, g$ are expressed by (5).

(b) For $\delta = -\frac{r^2}{25}$, another set of Weierstrass elliptic function solutions is found:

$$u = \frac{1}{100qk\chi_x\varphi_y}\Big(-25qk\chi_{xxx}\varphi_y - 25q\chi_x\varphi_t + 25pq^2k\chi_x^3\varphi_y \pm 30\sqrt{-6pq}kr\chi_x\chi_{xx}\varphi_y f(\xi) - 75pqkr\chi_x^3\varphi_y f(\xi) + 72pkr^2\chi_x^3\varphi_y f^2(\xi) - 75pqk\chi_x\chi_{xx}\varphi_y g(\xi) \pm 30\sqrt{-6pq}pkr\chi_x^3\varphi_y f(\xi)g(\xi)\Big),$$

$$v = \frac{3}{100q}pr\chi_x\varphi_y f(\xi)(-25q + 24rf(\xi) \pm 10\sqrt{-6pq}g(\xi)), \tag{15}$$

where $p = \pm 1$; $q$ and $r$ in (14) and (15) are arbitrary constants, $f$ and $g$ are expressed by (6).

(c) For $\delta = h^2 - s^2$ and $pq < 0$, a couple of solitary solutions is

$$u = \frac{1}{4qk\chi_x\varphi_y}\Big(-qk\chi_{xxx}\varphi_y - q\chi_x\varphi_t + pq^2k\chi_x^3\varphi_y \pm 3\sqrt{(s^2 - h^2 - r^2)pq}k\chi_x\chi_{xx}\varphi_y f(\xi) - 3pqrk\chi_x^3\varphi_y f(\xi) - 3pk\varphi_y(s^2 - h^2 - r^2)\chi_x^3 f^2(\xi) - 3pqk\chi_x\chi_{xx}\varphi_y g(\xi) \pm 3\sqrt{(s^2 - h^2 - r^2)pq}pk\chi_x^3\varphi_y f(\xi)g(\xi)\Big),$$

$$v = \frac{3}{4q}p\chi_x\varphi_y f(\xi)(-qr - (s^2 - h^2 - r^2)f(\xi) \pm \sqrt{(s^2 - h^2 - r^2)pq}g(\xi)), \tag{16}$$

where $p = \pm 1$; $s$ and $h$ are arbitrary constants, $f$ and $g$ are expressed by (7).

(d) For $\delta = -h^2 - s^2$ and $pq > 0$, the trigonometric function solutions are

$$u=\frac{1}{4qk\chi_x\varphi_y}\left(-qk\chi_{xxx}\varphi_y-q\chi_x\varphi_t+pq^2k\chi_x^3\varphi_y\pm 3\sqrt{(s^2+h^2-r^2)pq}k\chi_x\chi_{xx}\varphi_y f(\xi)-3pqkr\chi_x^3\varphi_y f(\xi)\right.$$
$$\left.-3pk(s^2+h^2-r^2)\chi_x^3\varphi_y f^2(\xi)-3pqk\chi_x\chi_{xx}\varphi_y g(\xi)\pm 3\sqrt{(s^2+h^2-r^2)pq}pk\chi_x^3\varphi_y f(\xi)g(\xi)\right), \quad (17)$$
$$v=\frac{3}{4q}p\chi_x\varphi_y f(\xi)\left(-qr-(s^2+h^2-r^2)f(\xi)\pm\sqrt{(s^2+h^2-r^2)pq}g(\xi)\right),$$

where $p=\pm 1$; $s$ and $h$ are arbitrary constants, $f$ and $g$ are expressed by (8).

(e) For $q=0$, a pair of rational solutions is

$$u=-\frac{1}{16k\chi_x\varphi_y}\left(4k\chi_{xxx}\varphi_y+4\chi_x\varphi_t+6pk\left(\pm\sqrt{C_1^2+4C_2pr}\chi_x\chi_{xx}-2r\chi_x^3\right)\varphi_y f(\xi)\right.$$
$$\left.+3p^2k(C_1^2+4C_2pr)\chi_x^3\varphi_y f^2(\xi)+12pk\chi_x\chi_{xx}\varphi_y g(\xi)\pm 6\sqrt{c_1^2+4c_2pr}p^2kH\chi_x^3 f(\xi)g(\xi)\right), \quad (18)$$
$$v=-\frac{3}{16}p\chi_x\varphi_y f(\xi)\left(4r+p(C_1^2+4C_2pr)f(\xi)\pm 2p\sqrt{C_1^2+4C_2pr}g(\xi)\right),$$

where $C_1, C_2$ and $r$ are arbitrary constants, $p=\pm 1$, $f$ and $g$ are expressed by (9).

### 3. New Type of Interaction between Solitons in the (2+1)-Dimensional BS Equation

In this section, we discuss a novel localized structure for the physical quantity $v$ expressed by (18).

For simplification, we take the constants $p=1, r=2, C_1=3$ and $C_2=4$, then the absolute value $V$ of the function $v$ in (18), henceforth denoted as physical quantity, becomes

$$V\equiv|v|=\frac{3}{4}\left|\chi_x\varphi_y\left(\frac{8}{2(\chi+\varphi)^2+3(\chi+\varphi)-4}+\frac{82-6\sqrt{41}-8\sqrt{41}(\chi+\varphi)}{(2(\chi+\varphi)^2+3(\chi+\varphi)-4)^2}\right)\right|, \quad (19)$$

where $\chi\equiv\chi(x)$ is an arbitrary function of $x$, $\varphi\equiv\varphi(y,t)=\varphi(Hy+St)$, $H$ and $S$ are two arbitrary constants. As the arbitrariness of the functions $\chi$ and $\varphi$ included in (19), the physical quantity $V$ may obviously extend to several different types of structures, such as the multi-soliton solutions, multi-dromion and dromion lattices solutions, multiple ring soliton solutions, breathers, instantons, oscillating solitons, peakons, compactons, chaotic and fractal localized solutions. For instance, a dromion solution which is localized in all directions can be driven by multiple straight-line and straight-line ghost solitons. When the functions $\chi$ and $\varphi$ are simply chosen as

$$\chi=\sum_{i=1}^{M}e^{K_i x+x_{0i}},$$

$$\varphi=1+\sum_{i=1}^{N}e^{k_i y+l_i t+y_{0i}},$$

where $K_i$, $k_i$, $l_i$, $x_{0i}$ and $y_{0i}$ are arbitrary constants and $M,N$ are positive integers, we can obtain a particular multi-dromion solution for the physical quantity $V$. Meanwhile, when the functions $\chi$ and $\varphi$ are considered to be some piecewise smooth functions, we can derive some multi-peakon excitations for the physical quantity $V$. For example, when

$$\chi=\sum_{i=1}^{M}\begin{cases}\chi_i(x), & x\le 0,\\ -\chi_i(-x)+2\chi_i(0), & x>0,\end{cases}$$

$$\varphi=\sum_{i=1}^{N}\begin{cases}\varphi_i(y+kt), & y+kt\le 0,\\ -\varphi_i(-y-kt)+2\varphi_i(0), & y+kt>0,\end{cases}$$

where the functions $\chi_i(x)$ and $\varphi_i(y+kt)\equiv\varphi(\xi)$ are the differentiable functions of the indicated arguments and possess the boundary conditions

$$\chi_i(\pm\infty)=A_{\pm i}, \quad (i=1,2,\cdots,M),$$
$$\varphi_i(\pm\infty)=B_{\pm i}, \quad (i=1,2,\cdots,N)$$

with $A_{\pm i}$ and $B_{\pm i}$ being constants and/or even infinity, we can derive a special multi-peakon solution for the physical quantity $V$. Since similar situations have been widely discussed in some previous publications [6–9, 16], the related plots are omitted in our present paper. Here we are interested in studying some new type of interaction between solitons in the (2+1)-dimensional BS equation.

Based on the physical quantity (19), semi-folded localized structures can be discussed, when the function $\chi$ is a single-valued function and $\varphi$ is selected via the relations

$$\begin{aligned} \varphi_y &= \sum_{i=1}^{N_1} \alpha_i(\zeta - d_i t), \\ y &= \zeta + \sum_{j=1}^{N_2} \beta_j(\zeta - e_j t), \\ \varphi &= \int^{\zeta} \varphi_y y_\zeta \mathrm{d}\zeta, \end{aligned} \tag{20}$$

where $d_i, e_j$ $(i = 1,2,\cdots,N_1, j = 1,2,\cdots,N_2)$ are arbitrary constants, $\alpha_i$ and $\beta_j$ are localized excitations with the properties $\alpha_i(\pm\infty) = 0, \beta_j(\pm\infty) =$ constant. From (20), one can show that $\zeta$ may be a multi-valued function in some suitable regions of $y$ by choosing the functions $\beta_j$ appropriately. Therefore, the function $\varphi_y$, which is obviously an interaction solution of $N_1$ localized excitations due to the property $\zeta|_{y\longrightarrow\infty} \longrightarrow \infty$, may be a multi-valued function of $y$ in these areas, though it is a single-valued function of $\zeta$. Actually, most of the known multi-loop solutions are special cases of (20).

### 3.1. A Compacton-Like Semi-Foldon Structure

Here we first discuss a new type of coherent structure for the physical quantity $V$, and focus our attention on a (2+1)-dimensional semi-folded localized structure, which may exist in certain intricate situations. When the function $\chi$ is a single-valued function and $\varphi$ is selected as (20) with $N_1 = N_2 = 1$, namely

$$\chi = 8 + \begin{cases} 0, & x \le -\dfrac{\pi}{2}, \\ 1 + \sin^4 x + \cos^4 x, & -\dfrac{\pi}{2} < x \le \dfrac{\pi}{2}, \\ 2, & x > \dfrac{\pi}{2}, \end{cases} \tag{21}$$

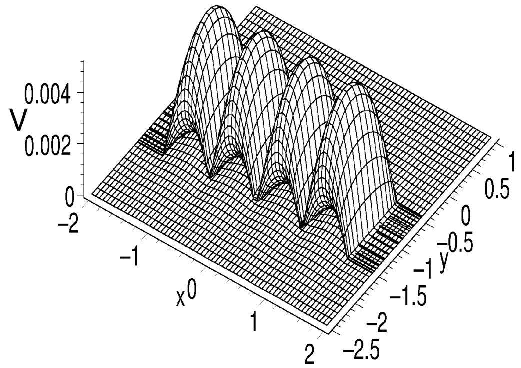



Fig. 1. Compacton-like semi-foldon structure for the physical quantity $V$ (19) with conditions (21) and (22) at time $t = -2$.

$$\begin{aligned} \varphi_y &= 0.5\mathrm{sech}^2(\zeta - 0.5t), \\ y &= \zeta - 2\tanh(\zeta - 0.5t), \\ \varphi &= \int^{\zeta} \varphi_y y_\zeta \mathrm{d}\zeta, \end{aligned} \tag{22}$$

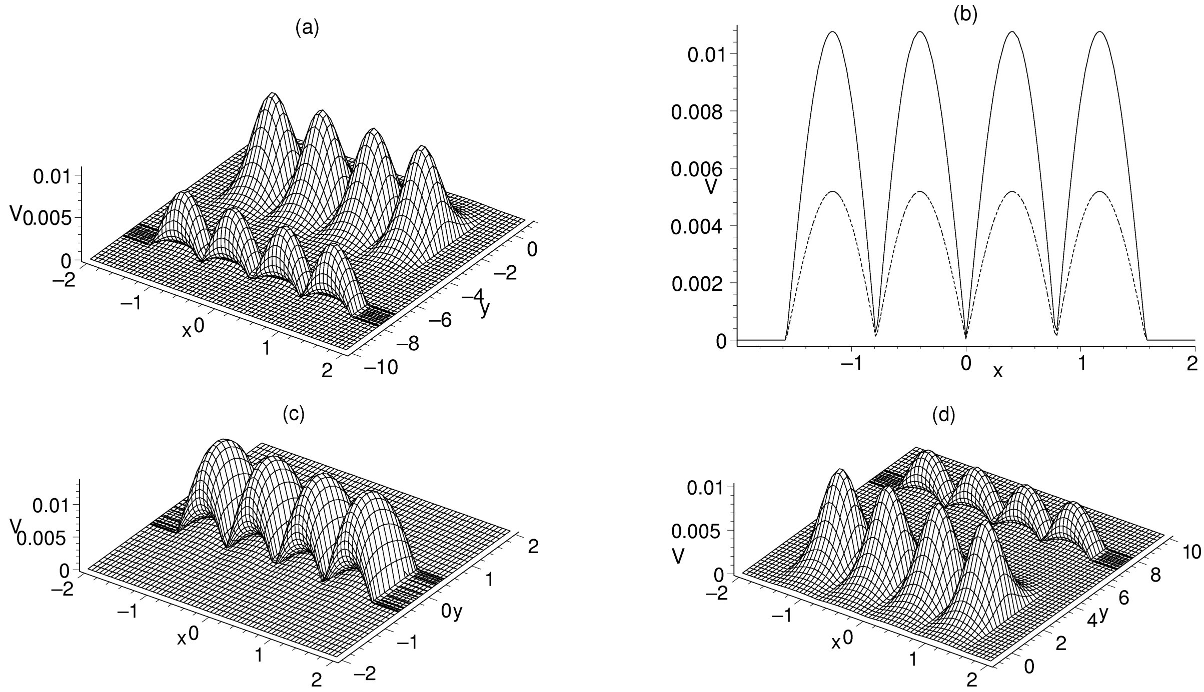
we can obtain a new type of semi-folded localized excitation which looks like a compacton-like loop soliton (compacton-like semi-foldon). The corresponding coherent structure is displayed in Fig. 1, from which one can find that the semi-folded coherent structure possesses a novel property, namely, it folds as a loop soliton in the $y$-direction but localizes as a 4-compacton in the $x$-direction.

### 3.2. Interaction between a Compacton-Like Semi-Foldon and a 4-Compacton

Interaction between the new solitons deserves further investigation. The interaction can be elastic or inelastic. It is called elastic, if the amplitude, velocity and wave shape of two solitons do not change after interaction. Fusion or fission of component solitons has also been observed to interact in an elastic or an inelastic way. Here, due to the arbitrariness of the functions in (19), we can discuss the interaction behavior between a compacton-like semi-foldon and a 4-compacton by selecting $\chi$ as a single-valued piecewise smooth function (21) and $\varphi$ as

$$\varphi_y = 0.5\mathrm{sech}^2(\zeta - 0.5t) + 0.9\mathrm{sech}^2\zeta, y = \zeta - 2\tanh(\zeta - 0.5t), \varphi = \int^{\zeta} \varphi_y y_\zeta \mathrm{d}\zeta. \tag{23}$$

From Fig. 2, we can see that the interaction between the multi-valued compacton-like semi-foldon and the



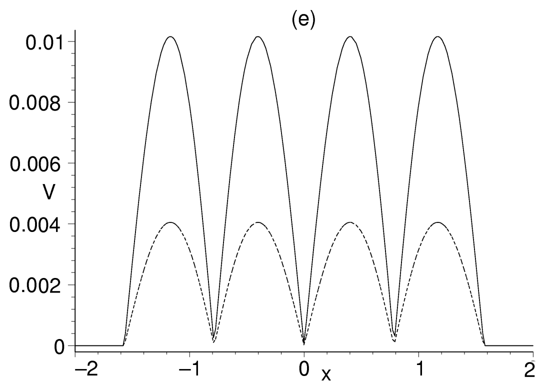




Fig. 2. Evolution profiles of the interaction between a compacton-like semi-foldon and a 4-compacton determined by (19) with conditions (21) and (23) at (a) $t = -15$, (c) $t = -0.1$, (d) $t = 15$. (b) The corresponding sectional view of (a) at $y = -2$ (solid line) and $y = -7.08$ (dotted line), respectively. (e) The corresponding sectional view of (d) at $y = 2$ (solid line) and $y = 7.08$ (dotted line), respectively.

single-valued compacton is inelastic since the amplitudes of the static compacton and the moving semi-foldon decrease. This property is different from the known completely elastic interaction between either single-valued and single-valued solitons [16], or multi-valued and multi-valued solitons (foldons) [9].

## 4. Summary and Discussion

In conclusion, using an auxiliary equation, we successfully obtain the variable separation solutions of the (2+1)-dimensional BS equation. From the special variable separation solution (18) and by selecting appropriate functions, a novel type of interaction between the multi-valued and the single-valued solitons is found. By means of the derived profiles, one finds that their interaction is inelastic, which is different from the completely elastic interaction between either single-valued and single-valued solitons, or multi-valued and multi-valued solitons (foldons).

Moreover, the work in this paper gives an example about how to use a direct and simple method to obtain variable separation solutions for other (2+1)-dimensional systems, since Zheng et al. [8, 9] applied the mapping method only for the BKK hierarchy. It is implied that this simple and direct method is an alternative method to realize variable separation for the (2+1)-

dimensional systems. This work is worthwhile further studying.

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